Logarithmic Velocity Profile for Turbulent Flow in Straight Rough Pipe and Evaluation of Karman Constant with Boundary Layer Reynolds Number- A Complete Solution

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Abstract- This paper deals with the derivation of logarithm velocity distribution in the turbulent overlap region considering roughness element *k* as an important parameter. Milikan's method primarily did not consider roughness element. Complete solution together with the value of Karman constant has been obtained. The mathematical structure is based on Ipsen's method of dimensional analysis followed by Boltzmann entropy formalism.

Index Terms— Karman-Prantdl velocity distribution for turbulent flow, Reynolds number, laminar sublayer.

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1 Introduction

Ludwig Prandtl is the father of modern fluid mechanics. He not only surpassed the uses of Navier-Stokes equation by his boundary layer theory (1904) but presented his mixing layer theory in 1925 advancing the concept of turbulent shear stress due to Osborne Reynolds (1886). Theodor Von Karman, student of Prandtl rederived the same law in 1930. Prandtl made two assumptions – shear stress in the turbulent boundary layer is constant and equals boundary shear stress (τ_ω) and the mixing length (l) i.e. the distance between two layers in the transverse direction such that particles from one can get into the other is directly proportional to distance (y) from boundary. Constant of proportionality k_r is called Karman constant. Karman on the other hand assumed [1] linear stress distribution together with l being directly proportional to $\left(du/dy\right)/\left(d^2u/dy^2\right)$. C.B. Millikan in 1938 adopted a dimensional method without going into physical detail of the process. He derived the logarithmic law on the consideration that the inner layer flow with flow parameters $u, y, \tau_\omega, \rho, \nu$ leading to the dimensional relation $u/u_* = f\left(u_* y/\nu\right)$ and outer layer flow with parameters $U, y, \tau_\omega, \rho, \delta$ leading to dimensional relation $\left(U - u_*\right)/u_* = g\left(y/\delta\right)$ can overlap smoothly if the velocity profile is logarithmic. So his method employs 6 parameters and out of which 5 are to be choosen at a time for two different layers. Roughness element *k* and its effects were not considered. In the following section a derivation based on Ipsen's (1960) method of dimensional analysis will be presented taking the roughness element k into account. This not only indicates the elegance of the method but also avoids any specific model of the process bringing out the essence of logarithmic velocity distribution for turbulent flow. The derivation shows that logarithmic velocity distribution is not an outcome of some choice or assumption but reflects the very nature of flow associated with boundary layer. Remarkable feature of the analysis is that it can estimate Karman constant theoretically.

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2 Prandtl-Karman Universal Velocity Distribution Law for Turbulent Boundary Overlap Layer with Boundary Texture

$$\frac{u}{u_*} = \frac{1}{k_r} ln\left(y/y'\right) \tag{1}$$

$$u_* = \sqrt{\tau_\omega/\rho} \tag{2}$$

Here u_* is called shear velocity and y' is some characteristic distance from the boundary where velocity (u) becomes zero. τ_ω represents wall shear stress due to viscous effect and roughness and ρ is the density of fluid. Several experiments estimated k_r to a value close to 0.41.

Nikuradse's experiments (1932; 1933) on smooth and rough pipe produced the following estimates on y' in terms of k (Nikuradse sand grain roughness) and thickness (δ_{ls}) of laminar sub layer. Nikuradse's k is actually mean diameter of sand particle. Here δ_{ls} is given by

$$\delta_{ls} = 11.6\nu/u_* \tag{3}$$

ν is the kinematic viscosity and is the ratio of coefficient of viscosity(η) to density(ρ).

Herman Schlichting Criterion (1955) Nikuradse's estimate($y^{/}$) of surface

Smooth surface:

$$k/\delta_{ls} < \frac{1}{2} \text{ i.e. } ku_*/\nu < 5 \qquad y^{/} = \delta_{ls}\Big/107 = \frac{1}{9.22}\frac{\nu}{u_*}$$

Transitional roughness:

$$0.5 < k/\delta_{ls} < 6 \text{ i.e. } 5 < ku_*/\nu < 70$$

Fully rough surface:

$$k/\delta_{ls} > 6 \text{ i.e. } ku_*/\nu > 70 \qquad y^{/} = k/30$$

Substituting these values of $y^{/}$ and k_r=0.41 in equation (1) we get

$$u/u_* = 2.44\,ln\left(u_* y/\nu\right) + 5.42 \quad \text{(smooth surface)} \tag{4a}$$

$$u/u_* = 2.44\,ln\left(y/k\right) + 8.3 \quad \text{(rough surface)} \tag{4b}$$

Equation (4a) is valid for $y/y^{/} > 280$ i.e. for $y/\delta_{ls} > 2.6$ or $yu_*/\nu > 30$. If $5 < yu_*/\nu < 30$ then the layer of flow is called buffer layer. In this layer turbulence and laminar motion coexists. Equation (4b) is valid for $y/y^{/} > 30$ i.e. for $y/k > 1$.

Equations 4(a) and 4(b) represent the Prandtl Karman one dimensional velocity distribution for smooth and rough boundaries in the turbulent region. If we use k_r = 0.4 we get convenient figures such as 2.5 instead of 2.44 and 5.5 and 8.5 for 5.42 and 8.3. Texts generally refer to these values.

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3 DIMENSIONAL DERIVATION

Close to the boundary but outside laminar sublayer we can choose the following functional form of u. Here u depends on both inertia parameter (ρ) and viscosity parameter (η).

$$u = f_1\left(\eta \quad ,\rho \quad ,\tau_\omega \quad ,k \quad , y \right) \tag{5a}$$

$$[LT^{-1}] \quad [ML^{-1}T^{-1}] \quad [ML^{-3}] \quad [ML^{-1}T^{-2}] \quad [L] \quad [L]$$

Ipsen suggested step by step method by successive elimination of M, L and T in successive steps through division or multiplication by suitable dependent variable with suitable power. Let's start with elimination of M.

$$u = f_2\left(\eta\big/\rho \quad ,\tau_\omega\big/\rho \quad ,k \quad ,y\right)$$

$$[LT^{-1}] \quad [L^2T^{-1}] \quad [L^2T^{-2}] \quad [L] \quad [L]$$

Here one thing which is very important to note that division by ρ eliminates M from those terms where it was and other terms are left undisturbed. Now let's rewrite the above equation

$$u = f_2\left(\nu \quad ,u_*^2 \quad ,k \quad ,y\right)$$

$$[LT^{-1}] \quad [L^2T^{-1}] \quad [L^2T^{-2}] \quad [L] \quad [L]$$

$$u = f_3\left(\nu \quad ,u_* \quad ,k \quad ,y\right)$$

$$[LT^{-1}] \quad [L^2T^{-1}] \quad [LT^{-2}] \quad [L] \quad [L]$$

$$\Downarrow \div u_* \quad \Downarrow \div u_*$$

$$\frac{u}{u_*} = f_4\left(\frac{\nu}{u_*} \quad ,k \quad ,y\right)$$

$$[M^0L^0T^0] \quad [L] \quad [L] \quad [L]$$

$$\Downarrow \div \frac{\nu}{u_*} \quad \Downarrow \div \frac{\nu}{u_*}$$

$$\frac{u}{u_*} = f_5\left(\frac{ku_*}{\nu},\frac{yu_*}{\nu}\right) \tag{5b}$$

This is the required non dimensional functional form close to boundary. We could also obtain the final form dividing by k for ν/u_* and that would yield

$$\frac{u}{u_*} = f^{/}\left(\frac{\nu}{ku_*},\frac{y}{k}\right) \tag{5c}$$

As numbers of primary dimensions are three (M, L, T) and number of independent variables are five so the non dimensional equation will contain 5-3=2 number of non dimensional parameters (π-term) by Bukingham's π-theorem. Equation (5b) or (5c) both contains same information and represents the same thing. Ipsen's method is remarkably easy and straight forward and simultaneously more elegant compared to Rayleigh's method or even to Bukingham's method. The power of dimensional analysis is noteworthy. Equation (4a) or (4b) as obtained from Prandtl-Karman equation (1) contains two dimensionless parameters yu^*/ν or y/k but for this we had to use Nikuradse's experimental result on $y^{/}$; but by dimensional approach we directly obtain these parameters in a natural way. We can even improve the knowledge about the functional form by physical reasoning. Equation (5c) is a more convenient form compared to (5b) because k i.e. roughness height when is large enough then the nondimensional term ν/ku^* will be small enough and the velocity depends only upon the term y/k. In fact large roughness height destroys the viscous laminar sublayer adjacent to the boundary and the turbulent velocity is

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fully governed by k and not by boundary viscous shear term ν/ku^* produced by roughness. For smooth boundary however the viscous parameter ν/ku^* contributes its effect only to the boundary layer and should in no way contribute at large distance y. With a view to this physical ground the viscous term ku^*/ν can in no way be considered entangled with the term y/k and the functional form obtained in equation (5c) can be separated as

$$\frac{u}{u_*} = F(\nu/ku_*) + f(y/k)$$

or
$$\frac{u}{u_*} - F(\nu/ku_*) = f(y/k) = f(\omega_1) \tag{5d}$$

If we now consider free stream velocity U close to central axis then the functional form of U, following equation (5a), can be written as

$$U = h_1(\eta, \rho, \tau_\omega, k, \delta_t) \tag{6a}$$

Here δ_t is the width of turbulent boundary layer. By same argument

$$\frac{U}{u_*} = h'(\nu/ku_*, \delta_t/k) \tag{6b}$$

or
$$\frac{U}{u_*} - H(\nu/ku_*) = h(\delta_t/k) = h(\omega) \tag{6c}$$

Now let us find the functional form of U-u. We can choose the functional form as

$$U - u = g_1(\rho, \tau_\omega, \delta_t, y) \tag{7a}$$

[LT-1] [ML-3] [ML-1T-2] [L] [L]

The difference U-u has been chosen independent of coefficient of viscosity η on the physical ground that the boundary layer viscous effects determined by η either will cancel out in the difference or will contribute little to the difference of U-u. Now we follow Ipsen's method once again starting with elimination of primary dimension M.

$$U - u = g_2(\tau_\omega/\rho, \delta_t, y) = g_3(u_*^2, \delta_t, y)$$

$$U - u = g_4(u_*, \delta_t, y)$$

[LT-1] [LT-1] [L] [L]

$\Downarrow \div u_*$ $\Downarrow \div u_*$

or
$$\frac{U-u}{u_*} = g_5(\delta_t, y)$$

$\Downarrow \div y$ $\Downarrow \div y$

or
$$\frac{U-u}{u_*} = g(\delta_t/y) = g(\omega_2) \tag{7b}$$

Now we are ready for evaluation of functional form of u. Let's concentrate on equation (5d), (6c) and (7b). We see the following relations

$$f(\omega_1) + g(\omega_2) = U/u_* - F(\nu/ku_*)$$
$$= h(\omega) + H(\nu/ku_*) - F(\nu/ku_*) \tag{8}$$

As $\omega_1 = y/k$, $\omega_2 = \delta_t/y$ and $\omega = \delta_t/k$ so $f(\omega_1) + g(\omega_2)$ must be independent of functional dependence on ν i.e. on ν/ku_*. This means H=F and finally we get

$$f(\omega_1) + g(\omega_2) = U/u_* - F(\nu/ku_*) = h(\omega) \tag{9a}$$

and
$$\omega_1\omega_2 = \omega \tag{9b}$$

If however we assume same functional form for u and U then the functional form h_1 given by equation (6a) is same as f_1 given by equation (5a) and for equation (6c) we directly get

$$\frac{U}{u_*} - F(\nu/ku_*) = f(\delta_t/k) = f(\omega)$$

So functional form $h(\omega)$ is same as $f(\omega)$.

The relations among ω_1, ω_2 and ω given by equation (9a) and (9b) are just like entropy versus thermodynamic probability relations between subsystems and system. Entropy relations are given by

$$s_1(\omega_1) + s_2(\omega_2) = s(\omega)$$
$$\omega_1\omega_2 = \omega$$

Here $s(\omega)$ is entropy of the system comprising of subsystems 1 and 2. Boltzmann obtained the functional form of $s = k\ln\omega + c$ from these two properties of s and ω. We now will adopt same method for equation (9a) and (9b).

$$\frac{\partial f(\omega_1)}{\partial \omega_1} + \frac{\partial g(\omega_2)}{\partial \omega_1} = \frac{\partial h(\omega)}{\partial \omega_1}$$

or,
$$\frac{\partial f(\omega_1)}{\partial \omega_1} = \frac{\partial h(\omega)}{\partial \omega}\frac{\partial \omega}{\partial \omega_1} = \omega_2\frac{\partial h(\omega)}{\partial \omega}$$

or,
$$\omega_1\frac{\partial f(\omega_1)}{\partial \omega_1} = \omega_1\omega_2\frac{\partial h(\omega)}{\partial \omega} = \omega\frac{\partial h(\omega)}{\partial \omega} = 1/k_r$$

Here k_r is constant and independent of ω_1 or ω i.e. independent of ω_2 also. From the above relation we can write

or,
$$df(\omega_1) = (1/k_r)\, d\omega_1/\omega_1$$

On integration the equation yields

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$$f(\omega_1) = \frac{1}{k_r} ln\, \omega_1 + c_1$$

c_1 is constant of integration and is independent of ω_1 . From equation (5d) we get

$$u/u_* = \frac{1}{k_r} ln(y/k) + c_1 + F(\nu/ku_*) \quad (10a)$$

From equation (10a) we get

$$U/u_* - F(\nu/ku_*) = \frac{1}{k_r} ln\frac{\delta_t}{k} + c_1 \,.$$

Now comparing with equation (8) we get

$$U/u_* - F(\nu/ku_*) = h(\omega) = \frac{1}{k_r} ln\frac{\delta_t}{k} + c_1 \quad (10b)$$

If we perform partial differentiation w.r.t. ω_2 on both sides of equation (9a) we get

$$\frac{\partial g(\omega_2)}{\partial \omega_2} = \omega_1 \frac{\partial h(\omega)}{\partial \omega}$$

$$\omega_2 \frac{\partial g(\omega_2)}{\partial \omega_2} = \omega \frac{\partial h(\omega)}{\partial \omega} = 1/k_r$$

Integrating above equation we have

$$g(\omega_2) = \frac{1}{k_r} ln\, \omega_2 + c_2$$

c_2 is constant of integration and is independent of ω_2 .

$$\frac{U-u}{u_*} = \frac{1}{k_r} ln(\delta_t/y) + c_2$$

If we put $u=U$ for $y=\delta_t$ we get c_2 =0. This means

$$\frac{u}{u_*} = \frac{1}{k_r} ln(\frac{y}{\delta_t}) + \frac{U}{u_*} \quad (10c)$$

This equation is an alternative form of equation (10a). From equation (10b) we get

$$c_1 = \frac{U}{u_*} - \frac{1}{k_r} ln(\frac{\delta_t}{k}) - F(\nu/ku_*) \quad (10d)$$

This is a remarkable result. It was impossible by Prandtl's mixing layer theory alone to find the expression of c_1. If we put $u = u_{ls}$ for $y = \delta_{ls}$ for smooth boundary where δ_{ls} thickness of laminar sublayer then using equation (10d) is and (10c) we get

$$k_r = \frac{u_*}{U-u_{ls}} ln(\delta_t/\delta_{ls})$$

i.e. $$k_r = \frac{1}{(U-u_{ls})/u_*} ln(\delta_t/\delta_{ls}) \quad (10e)$$

So we can find also the value of Karman constant theoretically. This programme will be resumed again in section 4.

Now substituting the value of c_1 from equation (10d) in equation (10b) we get

$$U/u_* - F(\nu/ku_*) = \frac{1}{k_r} ln\frac{\nu}{ku_*} + U/u_*$$

or, $$F(\nu/ku_*) = \frac{1}{k_r} ln\frac{ku_*}{\nu}$$

Therefore for smooth surfaces close to which laminar sublayer exists we finally get from equation (10a)

$$u/u_* = \frac{1}{k_r} ln(yu_*/\nu) + c_1 \quad (10f)$$

Equation (10f) is exactly the same equation as (4a) for smooth surface. It is an astonishing result. Without going into physical details of velocity distribution across the stream one can obtain the same result only from dimensional analysis coupled with physical reasoning. It means that even without the model of the exact physical process we can go far based on some general guidelines. If however we think of very rough boundary then laminar sublayer is completely destroyed and value of

$F(\nu/ku_*)$ is never $\frac{1}{k_r} ln\frac{ku_*}{\nu}$.

The full solutions are therefore given in the following.

$$u/u_* = \frac{1}{k_r} ln(yu_*/\nu) + c_1 \quad \text{(smooth boundary)} \quad (10f)$$

c_1 is constant and is independent of y/k . k_r is Karman constant and is independent of y/k , δ_t/y and δ_t/k . It is however possible that c_1 and k_r may depend on u_* and U.

$$u/u_* = \frac{1}{k_r} ln(y/k) + c_1 + F(\nu/ku_*) \quad \text{(rough wall)} \quad (10a)$$

$F(\nu/ku_*) \rightarrow \frac{1}{k_r} ln(ku_*/\nu)$ for small values of ku_*/ν .

4 EVALUATION OF c_1 AND k_r IN TERMS OF COEFFICIENT OF FRICTION c_f, REYNOLDS NUMBER R_f AND FOR TURBULENT FLOW AND SHEAR REYNOLDS NUMBER R_{ls}^*

$$c_f = \tau_\omega / (\frac{1}{2}\rho U^2) = 2(u_*/U)^2$$

or, $$U/u_* = \sqrt{2/c_f} \quad (11)$$

From equation (10e) we get

$$k_r = \frac{1}{(U-u_{ls})/u_*} ln(\frac{u_*\delta_t/\nu}{u_*\delta_{ls}/\nu})$$

$$= \frac{1}{(U-u_{ls})/u_*} ln[\frac{(U\delta_t/\nu)(u_*/U)}{R^*_{ls}}]$$

$$= \frac{1}{\sqrt{\frac{2}{c_f}} - \frac{u_{ls}}{u_*}} ln[\frac{R_t\sqrt{c_f/2}}{R^*_{ls}}] \quad (12)$$

If we now apply equation (10f) for laminar sublayer top we get

$$c_1 = u_{ls}/u_* - \frac{1}{k_r} ln\frac{u_*\delta_{ls}}{\nu}$$

$$c_1 = u_{ls}/u_* - \frac{1}{k_r} lnR^*_{ls} \quad (13)$$

Although we can theoretically find the expression of k_r and c_1 from equation (12) and (13) but it is very difficult to find a general expression of c_f in terms of R_t . So we will use Nikuradse's estimates of $y^/$ (value of y close to boundary for which u=0) and δ_{ls} for smooth boundary.

$$\delta_{ls} = 11.6\nu/u_* \quad (14)$$

$$y^/ = \delta_{ls}/107 = \frac{1}{9.22}\frac{\nu}{u_*} \quad (15)$$

Substituting u=0 for $y=y^/$ in equation 10(f) we get

$$c_1 = 2.22/k_r \quad (16)$$

Boundary shear stress $\tau_\omega = u_*^2\rho$. In laminar sublayer we can write

$$\tau_\omega = \eta\, du/dy$$

or, $$\eta/\rho \frac{du}{dy} = u_*^2$$

Integration of the equation leads to

$$u = \frac{yu_*^2}{\nu} + A$$

At $y=y^/$ given by equation (15), u=0. So the above equation gives $A = -u_*/9.22$. So in laminar sublayer we have

$$\frac{u}{u_*} = \frac{yu_*}{\nu} - \frac{1}{9.22} \quad (17)$$

If we put $y = \delta_{ls}$ and $u = u_{ls}$ then from equation (17) we have

$$u_{ls}/u_* = \frac{u_*\delta_{ls}}{\nu} - \frac{1}{9.22} = R^*_{ls} - 1/9.22$$

R^*_{ls} is shear Reynolds number. As $R^*_{ls} = u_*\delta_{ls}/\nu = 11.6$ so we have,

$$u_{ls}/u_* = R^*_{ls} - 1/9.22 \approx 11.5 \quad (18)$$

From equation (13) we get

$$c_1 = u_{ls}/u_* - \frac{1}{k_r} ln\frac{u_*\delta_{ls}}{\nu}$$

$$= 11.5 - \frac{1}{k_r} ln 11.6$$

$$c_1 = 11.5 - 2.45/k_r \quad (19)$$

Now comparing equation (16) and (19) we get

$$2.22/k_r = 11.5 - 2.45/k_r$$

$$1/k_r = \frac{11.5}{4.67} = 2.46$$

or, $$k_r = 0.406 \approx 0.41 \quad (20)$$

and $$c_1 = 2.22/k_r = 5.47 \quad (21)$$

This is a remarkable result. Values of k_r and c_1 are too good in agreement with the values presented in equation 4(a).

Applying again Nikuradse's estimate $y^/ = k/30$ (value of $y^/$ for which u reduces to zero near rough boundary) to equation (10a) we get

$$1/k_r\, ln(1/30) + c_1 + F(\nu/ku_*) = 0$$

Substituting c_1=5.47 and k_r =0.406 we get $F(\nu/ku_*) = 2.91$.

So equation (10a) for rough boundary reduces to

$$\frac{u}{u_*} = \frac{1}{k_r} ln(y/k) + 8.38$$

Above equation is the same as given in 4(b). So we get complete solution for turbulent velocity distribution for rough and smooth straight pipe.

If we take partial derivative of equation (10a) with respect to y we get

$$\frac{1}{u_*}\frac{\partial u}{\partial y} = \frac{1}{k_r y}$$

or, $$u_*^2 = k_r^2 y^2 (\frac{\partial u}{\partial y})^2$$

or, $$\tau_\omega = \rho k_r^2 y^2 (\frac{\partial u}{\partial y})^2$$

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Above expression of turbulent boundary shear stress was the starting point of Prandtl-Karman mixing layer theory.

5 EVALUATION OF REYNOLDS NUMBER R_{Ls} AT THE TOP OF LAMINAR SUBLAYER

If we write $c_{fl} = \tau_{\omega} \Big/ (\frac{1}{2}\rho u_{ls}^2)$ then substituting

$\tau_{\omega} = \eta u_{ls} \big/ \delta_{ls}$ in the expression we have

$$c_{fl} = 2\nu \big/ u_{ls}\delta_{ls} = 2 \big/ R_{ls} \tag{22}$$

again $$c_{fl} = \tau_{\omega} \Big/ (\frac{1}{2}\rho u_{ls}^2) = 2\,(u_{*} \big/ u_{ls})^2 \tag{23}$$

Now substituting the expression of $u_{ls} \big/ u_{*}$ from equation (18) in equation (23) and then comparing with equation (22) we get

$$2 \big/ R_{ls} = 2 \big/ (R_{ls}^{*} - 1/9.22)^2$$

or, $$R_{ls} = (R_{ls}^{*} - 1/9.22)^2 = (11.5)^2 \approx 132 \tag{24}$$

This means that characteristic turbulent flow necessarily associated with eddies are set up close to boundary of pipe line where Reynolds number exceeds 130. Vortices start separating from the boundary layer and produce 'Karman Vortex sheet' at about $R \approx 100$ for flow past cylinder [2]. The result obtained in equation (24) is in good agreement with experiment.

6 CONCLUDING REMARKS

The treatment in this paper did not consider the effect of pressure gradient and acceleration due to gravity g. If these two parameters were included the number of independent dimensionless parameters (π terms) would be 4. In such situation the expression of velocity would have been more involved with the parameters. For equation (5c) we then had

$$\frac{u}{u_*} = f'\left(\frac{\nu}{yu_*}, \frac{k}{y}, \frac{u_*^2}{gy}, \frac{y}{\rho u_*^2}\frac{\partial p}{\partial x}\right)$$

subject to boundary condition that u=0 for y=y'. And x is distance along the central line of pipe. Similarly for equation (6b) we get

$$\frac{U}{u_*} = h'\left(\frac{\nu}{\delta_t u_*}, \frac{k}{\delta_t}, \frac{u_*^2}{g\delta_t}, \frac{y}{\rho u_*^2}\frac{\partial p}{\partial x}\right)$$

However the logarithmic turbulent velocity distribution in absence of these two extra parameters has been derived along with all constants in the present article.

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